THE DEFINING RELATIONS OF GEOMETRIC ALGEBRAS OF TYPE EC

AYAKO ITABA AND MASAKI MATSUNO

Abstract. In this report, we study geometric algebras of Type EC (these algebras are constructed from an elliptic curve in the projective plane). By using the defining relations of 3-dimensional Sklyanin algebras, we determine the defining relations of geometric algebras of Type EC. Also, we give criterion to check when two geometric algebras of Type EC are graded algebra isomorphic and graded Morita equivalent.

1. Geometric algebras

Through this report, let k be an algebraically closed field of characteristic 0, A a graded k-algebra finitely generated in degree 1. That is, $A = k\langle x_1, \cdots, x_n\rangle/I$, where $\deg x_i = 1$ for any $i = 1, \cdots, n$, and I is a homogeneous two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$ with $I_0 = I_1 = 0$. We call $A = k\langle x_1, \cdots, x_n\rangle/I$ a *quadratic algebra* if I is an ideal of $k\langle x_1, \cdots, x_n\rangle$ generated by homogeneous polynomials of degree two. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules. Morphisms in $\operatorname{GrMod} A$ are right A-module homomorphisms preserving degrees. We say that two graded algebras A and A' are *graded Morita equivalent* if the categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$ are equivalent. We denote by $\mathbb{P}^{n-1}$ the $n-1$ dimensional projective space over k.

For a quadratic algebra $A = k\langle x_1, \cdots, x_n\rangle/I$, we set

$$\Gamma_A := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0 \text{ for all } f \in I_2\},$$

where, for points $p = (p_1 : \cdots : p_n)$, $q = (q_1 : \cdots : q_n) \in \mathbb{P}^{n-1}$ and a homogeneous polynomial $f = \sum_{i,j} \alpha_{i,j} x_i x_j$ of degree two, we define $f(p, q) := \sum_{i,j} \alpha_{i,j} p_i q_j$.

A notion of *geometric algebra* was introduced by Mori [6].

Definition 1 ([6])**.** Let $A = k\langle x_1, \cdots, x_n\rangle/I$ be a quadratic algebra.

(1) We say that A satisfies (G1) if there exists a pair (E, σ), where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut}_k E$, such that

$$\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$, called the *geometric pair* of A.

(2) We say that A satisfies (G2) if there exists a pair (E, σ), where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut}_k E$, such that

$$I_2 = \{f \in k\langle x_1, \cdots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \text{ for all } p \in E\}.$$

In this case, we write $A = \mathcal{A}(E, \sigma)$.

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(3) A is called *geometric* if A satisfies both (G1) and (G2), and $A = \mathcal{A}(\mathcal{P}(A))$.

Note that, if A satisfies (G1), then A determines the pair (E, σ) by using Γ_A. Conversely, if A satisfies (G2), then A is determined by the pair (E, σ).

The following theorem tells us that classifying geometric algebras is equivalent to classifying their geometric pairs.

Theorem 2 ([6]). *Let A and A' be geometric algebras with $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A') = (E', \sigma')$ where E and E' are closed k-subschemes of $\mathbb{P}^{n-1}$, $\sigma \in \operatorname{Aut}_k E$ and $\sigma' \in \operatorname{Aut}_k E'$.*

(1) *$A \cong A'$ if and only if there exists $\tau \in \operatorname{Aut}_k \mathbb{P}^{n-1}$ which restricts to an isomorphism from E to E' such that the following diagram commutes*:

$$\begin{tikzcd} E \arrow[r, "\tau"] \arrow[d, "\sigma"'] & E' \arrow[d, "\sigma'"] \\ E \arrow[r, "\tau"] & E' \end{tikzcd}$$

(2) *A and A' are graded Morita equivalent if and only if there exists a sequence $\{\tau_i\}_{i \in \mathbb{Z}}$ where $\tau_i \in \operatorname{Aut}_k \mathbb{P}^{n-1}$ restricts to an isomorphism from E to E' for every $i \in \mathbb{Z}$ such that the following diagrams commute for all $i \in \mathbb{Z}$*:

$$\begin{tikzcd} E \arrow[r, "\tau_i"] \arrow[d, "\sigma"'] & E' \arrow[d, "\sigma'"] \\ E \arrow[r, "\tau_{i+1}"] & E' \end{tikzcd}$$

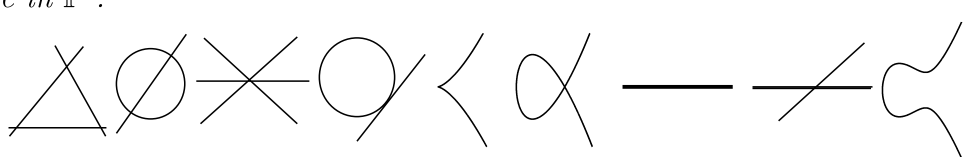
Artin and Schelter [1] defined a class of regular algebras which are main objects of study in noncommutative algebraic geometry.

Definition 3 ([1]). Let A be a noetherian connected graded k-algebra. A is called a *d-dimensional Artin-Schelter regular* (simply *AS-regular*) *algebra* if A satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,

(ii) (*Gorenstein condition*) $\quad \operatorname{Ext}^i_A(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

Theorem 4 ([2]). *Every 3-dimensional quadratic AS-regular algebra A is geometric. Moreover, when we write $\mathcal{P}(A) = (E, \sigma)$, E is either the projective plane $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$.*



(elliptic curve)

By the above theorem, it is important to study geometric algebras $\mathcal{A}(E, \sigma)$ when E is a cubic curve in $\mathbb{P}^2$. In this report, we focus on studying geometric algebras $\mathcal{A}(E, \sigma)$ when E is an elliptic curve. Such algebras are called *geometric algebras of Type EC*.

2. Defining relations of geometric algebras of Type EC

Let E be an elliptic curve in $\mathbb{P}^2$. In this section, we determine the automorphism group $\operatorname{Aut}_k E$ and compute the defining relations of geometric algebras $\mathcal{A}(E,\sigma)$ of Type EC.

It is well-known that the j-invariant $j(E)$ of E classifies elliptic curves in $\mathbb{P}^2$ up to isomorphism, that is, two elliptic curves E and E' are isomorphic if and only if $j(E) = j(E')$ (see [4]).

For an elliptic curve E in $\mathbb{P}^2$ and a point $p \in E$, we set

$$\operatorname{Aut}_k(E,p) := \{\sigma \in \operatorname{Aut}_k E \mid \sigma(p) = p\}.$$

It is known that, for every point $p \in E$, $\operatorname{Aut}_k(E,p)$ is a cyclic group and

$$|\operatorname{Aut}_k(E,p)| = \begin{cases} 2 & \text{if } j(E) \neq 0, 12^3, \\ 6 & \text{if } j(E) = 0, \\ 4 & \text{if } j(E) = 12^3 \end{cases}$$

(see [4]).

For each point $o \in E$, we can define an addition on E so that E is an abelian group with the zero element o and, for $p \in E$, the map σ_p defined by $\sigma_p(q) := p + q$ is a scheme automorphism of E, called the *translation* by a point p.

In this report, we use a *Hesse form* $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ where $\lambda \in k$ with $\lambda^3 \neq 1$. It is known that every elliptic curve in $\mathbb{P}^2$ can be written by a Hesse form (see [3]). For points $p = (a : b : c)$, $q = (\alpha : \beta : \gamma) \in E$, if we define

$$p + q := (ac\beta^2 - b^2\alpha\gamma : bc\alpha^2 - a^2\beta\gamma : ab\gamma^2 - c^2\alpha\beta),$$

then E is an abelian group with the zero element $0_E := (1 : -1 : 0) \in E$ (see [3]). From now on, we fix this group structure on E.

For an elliptic curve $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ in $\mathbb{P}^2$, the j-invariant is given by the formula

$$j(E) = \frac{27\lambda^3(\lambda^3 + 8)^3}{(\lambda^3 - 1)^3}$$

(see [3]).

If E and E' are elliptic curves in $\mathbb{P}^2$ with $j(E) = j(E')$, then there exists $\phi \in \operatorname{Aut}_k \mathbb{P}^2$ which restricts to an isomorphism from E to E'. For $\sigma \in \operatorname{Aut}_k E$, if we set $\sigma' := \phi\sigma\phi^{-1} \in \operatorname{Aut}_k E'$, then two geometric algebras $A = \mathcal{A}(E,\sigma)$ and $A' = \mathcal{A}(E',\sigma')$ are isomorphic by Theorem 2 (1). By this reason, when we study geometric algebras $\mathcal{A}(E,\sigma)$ of Type EC such that $j(E) = 0$, then we may assume $\lambda = 0$. A similar comment applies to the case $j(E) = 12^3$.

By using a Hesse form, we can give a generator of $\operatorname{Aut}_k(E, 0_E)$.

Lemma 5 ([5]). *Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$. A generator τ of $\operatorname{Aut}_k(E, 0_E)$ is given by*

$$\begin{cases} \tau(a : b : c) := (b : a : c) & \text{if } j(E) \neq 0, 12^3, \\ \tau(a : b : c) := (b : a : c\varepsilon) & \text{if } \lambda = 0 \text{ so that } j(E) = 0, \\ \tau(a : b : c) := (a\varepsilon^2 + b\varepsilon + c : a\varepsilon + b\varepsilon^2 + c : a + b + c) & \text{if } \lambda = 1 + \sqrt{3} \text{ so that } j(E) = 12^3 \end{cases}$$

for $(a : b : c) \in E$, where ε is a primitive 3rd root of unity.

From now on, we fix these generators. We set

$$\mathrm{Aut}_k\,(\mathbb{P}^2, E) := \{\sigma \in \mathrm{Aut}_k\,\mathbb{P}^2 \mid \sigma|_E \in \mathrm{Aut}_k\,E\}.$$

It follows from Lemma 5 that

$$\mathrm{Aut}_k\,(E, 0_E) \le \mathrm{Aut}_k\,(\mathbb{P}^2, E).$$

We set $T := \{\sigma_p \in \mathrm{Aut}_k\,E \mid p \in E\}$. It follows that

$$\mathrm{Aut}_k\,E \cong T \rtimes \mathrm{Aut}_k\,(E, 0_E) = \{\sigma_p\tau^i \in \mathrm{Aut}_k\,E \mid p \in E, i \in \mathbb{Z}_{|\tau|}\}$$

where

$$|\tau| = \begin{cases} 2 & \text{if } j(E) \neq 0, 12^3, \\ 6 & \text{if } \lambda = 0, \\ 4 & \text{if } \lambda = 1 + \sqrt{3}. \end{cases}$$

We call $p \in E$ *a 3-torsion point* if $3p = 0_E$ and set $E[3] := \{p \in E \mid 3p = 0_E\}$.

Lemma 6 ([5])**.** *Let E be an elliptic curve in $\mathbb{P}^2$, $p \in E$ and $i \in \mathbb{Z}_{|\tau|}$. A quadratic algebra $\mathcal{A}(E, \sigma_p\tau^i)$ satisfying* (G2) *is a geometric algebra of Type EC if and only if $p \in E \backslash E[3]$.*

Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$ and $p = (a : b : c) \in E\backslash E[3]$. If $\sigma = \sigma_p \in T$, then

$$A = \mathcal{A}(E, \sigma_p) = k\langle x, y, z\rangle / \begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix}.$$

This algebra $A = \mathcal{A}(E, \sigma_p)$ is called a 3*-dimensional Sklyanin algebra*.

If $\sigma = \sigma_p\tau^i \in \mathrm{Aut}_k\,E$, then we can compute the defining relations of $\mathcal{A}(E, \sigma)$ by using the defing relations of 3-dimensional Sklyanin algebras and $\tau^i \in \mathrm{Aut}_k\,(\mathbb{P}^2, E)$.

Theorem 7 ([5])**.** *Every geometric algebra $\mathcal{A}(E, \sigma)$ of Type EC is isomorphic to one of the following algebras $k\langle x, y, z\rangle/(f_1, f_2, f_3)$.*

(1) *If $j(E) \neq 0, 12^3$, then*

$$\begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases} \qquad \begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases}$$

where $(a : b : c) \in E\backslash E[3]$ and $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$.

(2) *If $j(E)=0$, then*

$$\begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases} \qquad \begin{cases} f_1 = axz + b\varepsilon zy + cyx, \\ f_2 = a\varepsilon zx + byz + cxy, \\ f_3 = ay^2 + bx^2 + c\varepsilon z^2. \end{cases}$$

$$\begin{cases} f_1 = ayz + b\varepsilon^2 zy + cx^2, \\ f_2 = a\varepsilon^2 zx + bxz + cy^2, \\ f_3 = axy + byx + c\varepsilon^2 z^2. \end{cases} \qquad \begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases}$$

$$\begin{cases} f_1 = ayz + b\varepsilon zy + cx^2, \\ f_2 = a\varepsilon zx + bxz + cy^2, \\ f_3 = axy + byx + c\varepsilon z^2. \end{cases} \qquad \begin{cases} f_1 = axz + b\varepsilon^2 zy + cyx, \\ f_2 = a\varepsilon^2 zx + byz + cxy, \\ f_3 = ay^2 + bx^2 + c\varepsilon^2 z^2. \end{cases}$$

where $(a:b:c) \in E \setminus E[3]$, $E = \mathcal{V}(x^3+y^3+z^3)$ and ε is a primitive 3rd root of unity.

(3) *If $j(E) = 12^3$, then*

$$\begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases} \quad \begin{cases} f_1 = a(\varepsilon x + \varepsilon^2 y + z)z + b(x+y+z)y + c(\varepsilon^2 x + \varepsilon y + z)x, \\ f_2 = a(x+y+z)x + b(\varepsilon^2 x + \varepsilon y + z)z + c(\varepsilon x + \varepsilon^2 y + z)y, \\ f_3 = a(\varepsilon^2 x + \varepsilon y + z)y + b(\varepsilon x + \varepsilon^2 y + z)x + c(x+y+z)z. \end{cases}$$

$$\begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases} \quad \begin{cases} f_1 = a(\varepsilon^2 x + \varepsilon y + z)z + b(x+y+z)y + c(\varepsilon x + \varepsilon^2 y + z)x, \\ f_2 = a(x+y+z)x + b(\varepsilon x + \varepsilon^2 y + z)z + c(\varepsilon^2 x + \varepsilon y + z)y, \\ f_3 = a(\varepsilon x + \varepsilon^2 y + z)y + b(\varepsilon^2 x + \varepsilon y + z)x + c(x+y+z)z. \end{cases}$$

where $(a:b:c) \in E \setminus E[3]$, $E = \mathcal{V}(x^3+y^3+z^3-3(1+\sqrt{3})xyz)$ and ε is a primitive 3rd root of unity.

3. Classification of geometric algebras of Type EC

In this section, we give criterion to check when two geometric algebras of Type EC are graded algebra isomorphic and graded Morita equivalent by using Theorem 2.

We set $T[3] := \{\sigma_p \in T \mid p \in E[3]\}$. It follows that $\operatorname{Aut}_k(\mathbb{P}^2, E) \cap T = T[3]$ (see [6]). Since $\operatorname{Aut}_k E \cong T \rtimes \operatorname{Aut}_k(E, 0_E)$ and $\operatorname{Aut}_k(E, 0_E) \le \operatorname{Aut}_k(\mathbb{P}^2, E)$, we can also determine the structure of $\operatorname{Aut}_k(\mathbb{P}^2, E)$.

Proposition 8 ([5])**.** *Let E be an elliptic curve in $\mathbb{P}^2$. Then*

$$\operatorname{Aut}_k(\mathbb{P}^2, E) \cong T[3] \rtimes \operatorname{Aut}_k(E, 0_E).$$

By Proposition 8, every automorphism $\sigma \in \operatorname{Aut}_k(\mathbb{P}^2, E)$ is written as $\sigma_p \tau^i$ where $p \in E[3]$ and $i \in \mathbb{Z}_{|\tau|}$. Hence we have the following results by Theorem 2.

Theorem 9 ([5])**.** *Let E be an elliptic curve in $\mathbb{P}^2$, $p, q \in E \setminus E[3]$ and $i, j \in \mathbb{Z}_{|\tau|}$.*

(1) $\mathcal{A}(E, \sigma_p\tau^i) \cong \mathcal{A}(E, \sigma_q\tau^j)$ *if and only if*
  (1-i) $i = j$, *and*
  (1-ii) *there exist $r \in E[3]$ and $l \in \mathbb{Z}_{|\tau|}$ such that $q = \tau^l(p) + r - \tau^i(r)$.*

(2) $\mathcal{A}(E, \sigma_p\tau^i)$ *and* $\mathcal{A}(E, \sigma_q\tau^j)$ *are graded Morita equivalent if and only if*
   (2-i) $p - \tau^{j-i}(p) \in E[3]$, *and*
   (2-ii) *there exist* $r \in E[3]$ *and* $l \in \mathbb{Z}_{|\tau|}$ *such that* $q = \tau^l(p) + r$.

Example 10. Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$. We assume that $j(E) \neq 0, 12^3$. Let $p = (a : b : c)$, $q = (b : a : c) \in E \setminus E[3]$ and let $A_1 = \mathcal{A}(E, \sigma_1)$, $A_2 = \mathcal{A}(E, \sigma_2)$, and $A_3 = \mathcal{A}(E, \sigma_3)$ where $\sigma_1 = \sigma_p$, $\sigma_2 = \sigma_q$ and $\sigma_3 = \sigma_p\tau$. The defining relations of A_1, A_2 and A_3 are given as follows:

$$A_1 = k\langle x, y, z\rangle / \begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix},$$
$$A_2 = k\langle x, y, z\rangle / \begin{pmatrix} byz + azy + cx^2 \\ bzx + axz + cy^2 \\ bxy + ayx + cz^2 \end{pmatrix},$$
$$A_3 = k\langle x, y, z\rangle / \begin{pmatrix} axz + bzy + cyx \\ azx + byz + cxy \\ ay^2 + bx^2 + cz^2 \end{pmatrix}.$$

It is not easy to see from their defining relations if A_1, A_2 and A_3 are graded algebra isomorphic or graded Morita equivalent, however, by using Theorem 9 we can determine if they are as follows:

(1) Since $\sigma_1 = \sigma_p\tau^0$ and $\sigma_2 = \sigma_q\tau^0$, the condition (1-i) is satisfied. Since $q = \tau(p) + r - \tau^0(r)$, the condition (1-ii) is also satisfied, hence $A_1 \cong A_2$ by Theorem 9 (1).
(2) Since $\sigma_1 = \sigma_p\tau^0$ and $\sigma_3 = \sigma_p\tau^1$, the condition (1-i) is not satisfied, so $A_1 \ncong A_3$.
(3) Since $p = \tau^0(p) + 0_E$, the condition (2-ii) is satisfied, so A_1 and A_3 are graded Morita equivalent if and only if the condition (2-i) is satisfied, that is, $p - \tau(p) \in E[3]$ by Theorem 9 (2). By calculation, we have $\tau(p) = -p$, so A_1 and A_3 are graded Morita equivalent if and only if $p \in E[6]$ where $E[6] := \{p \in E \mid 6p = 0_E\}$.

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. Math. **66** (1987), 171–216.
[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift Vol. **1** Birkhauser, Boston M.A., (1990), 33–85.
[3] H. R. Frium, *The group law on elliptic curves on Hesse form, Finite fields with applications to coding theory*, cryptography and related areas (Oaxaca, 2001), Springer, Berlin, (2002), 123–151.
[4] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, No. **52**, Springer-Verlag, New York-Heidelberg, (1977).
[5] A. Itaba and M. Matsuno, *Defining relations of* 3*-dimensional quadratic AS-regular algebras*, (2018) submitted (arXiv:1806.04940).
[6] I. Mori, *Non commutative projective schemes and point schemes*, Algebras, Rings and Their Representations, World Sci., Hackensack, N.J., (2006), 215–239.

Ayako Itaba
Department of Mathematics
Faculty of Science
Tokyo University of Science
1-3 Kagurazaka, Shinjyuku, Tokyo, 162-8601 JAPAN
E-mail address: itaba@rs.tus.ac.jp

Masaki Matsuno
Graduate School of Integrated Science and Technology
Shizuoka University
Ohya 836, Shizuoka, 422-8529 JAPAN
E-mail address: matsuno.masaki.14@shizuoka.ac.jp